Spring 2012

EE 330

ENGINEERING ELECTROMAGNETICS

HW 12: *Due Friday 20 April*

8.18, 8.28, 8.30, 8.43, 8.44, 9.2, 9.4, 9.5, 9.11, 9.15

Problem 8.18 For some types of glass, the index of refraction varies with wavelength. A prism made of a material with

$$n = 1.71 - \frac{4}{30}\lambda_0 \quad (\lambda_0 \text{ in } \mu\text{m}),$$

where λ_0 is the wavelength in vacuum, was used to disperse white light as shown in Fig. P8.18. The white light is incident at an angle of 50°, the wavelength λ_0 of red light is 0.7 μm, and that of violet light is 0.4 μm. Determine the angular dispersion in degrees.

Solution:

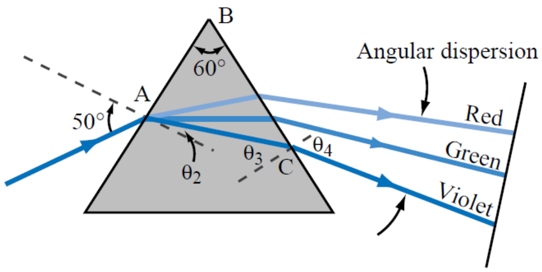

Figure P8.18: Prism of Problem 8.18.

For violet,

$$n_v = 1.71 - \frac{4}{30} \times 0.4 = 1.66, \qquad \sin\theta_2 = \frac{\sin\theta}{n_v} = \frac{\sin 50^\circ}{1.66},$$

or

$$\theta_2 = 27.48^\circ.$$

From the geometry of triangle ABC,

$$180^\circ = 60^\circ + (90^\circ - \theta_2) + (90^\circ - \theta_3),$$

or

$$\theta_3 = 60^\circ - \theta_2 = 60 - 27.48^\circ = 32.52^\circ,$$

and

$$\sin\theta_4 = n_v \sin\theta_3 = 1.66 \sin 32.52^\circ = 0.89,$$

or

$$\theta_4 = 63.18^\circ.$$

For red,

$$n_r = 1.71 - \frac{4}{30} \times 0.7 = 1.62,$$
$$\theta_2 = \sin^{-1}\left[\frac{\sin 50^\circ}{1.62}\right] = 28.22^\circ,$$
$$\theta_3 = 60^\circ - 28.22^\circ = 31.78^\circ,$$
$$\theta_4 = \sin^{-1}[1.62 \sin 31.78^\circ] = 58.56^\circ.$$

Hence, angular dispersion $= 63.18^\circ - 58.56^\circ = 4.62^\circ$.

Problem 8.28 Repeat Problem 8.27 for a wave in air with

$$\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)} \quad \text{(A/m)}$$

incident upon the planar boundary of a dielectric medium ($z \geq 0$) with $\varepsilon_{\mathrm{r}} = 9$.

Solution:

(a) $\widetilde{\mathbf{H}}^{\mathrm{i}} = \hat{\mathbf{y}}\, 2 \times 10^{-2} e^{-j(8x+6z)}$.

Since $\mathbf{H}^{\mathrm{i}}$ is along $\hat{\mathbf{y}}$, which is perpendicular to the plane of incidence, the wave is TM polarized, or equivalently, its electric field vector is parallel polarized (parallel to the plane of incidence).

(b) From Eq. (8.65b), the argument of the exponential is

$$-jk_1(x \sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) = -j(8x+6z).$$

Hence,

$$k_1 \sin\theta_{\mathrm{i}} = 8, \qquad k_1 \cos\theta_{\mathrm{i}} = 6,$$

from which we determine

$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{8}{6}\right) = 53.13^\circ,$$

$$k_1 = \sqrt{6^2 + 8^2} = 10 \quad \text{(rad/m)}.$$

Also,

$$\omega = u_{\mathrm{p}} k = ck = 3 \times 10^8 \times 10 = 3 \times 10^9 \quad \text{(rad/s)}.$$

(c)

$$\eta_1 = \eta_0 = 377\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{\eta_0}{3} = 125.67\ \Omega,$$

$$\theta_{\mathrm{t}} = \sin^{-1}\left[\frac{\sin\theta_{\mathrm{i}}}{\sqrt{\varepsilon_{\mathrm{r}_2}}}\right] = \sin^{-1}\left[\frac{\sin 53.13^\circ}{\sqrt{9}}\right] = 15.47^\circ,$$

$$\Gamma_{\parallel} = \frac{\eta_2\cos\theta_{\mathrm{t}} - \eta_1\cos\theta_{\mathrm{i}}}{\eta_2\cos\theta_{\mathrm{t}} + \eta_1\cos\theta_{\mathrm{i}}} = -0.30,$$

$$\tau_{\parallel} = (1 + \Gamma_{\parallel})\frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}} = 0.44.$$

In accordance with Eqs. (8.65a) to (8.65d), $E_0^{\mathrm{i}} = 2 \times 10^{-2}\eta_1$ and

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}}\cos\theta_{\mathrm{i}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{i}})\, 2 \times 10^{-2}\eta_1\, e^{-j(8x+6z)} = (\hat{\mathbf{x}}4.52 - \hat{\mathbf{z}}6.03)\, e^{-j(8x+6z)}.$$

$\widetilde{\mathbf{E}}^{\mathrm{r}}$ is similar to $\widetilde{\mathbf{E}}^{\mathrm{i}}$ except for reversal of z-components and multiplication of amplitude by $\Gamma_{\|}$. Hence, with $\Gamma_{\|} = -0.30$,

$$\mathbf{E}^{\mathrm{r}} = \mathfrak{Re}[\widetilde{\mathbf{E}}^{\mathrm{r}} e^{j\omega t}] = -(\hat{\mathbf{x}}\,1.36 + \hat{\mathbf{z}}\,1.81)\cos(3\times 10^9 t - 8x + 6z) \text{ V/m},$$

$$\begin{aligned}\mathbf{H}^{\mathrm{r}} &= \hat{\mathbf{y}}\,2\times 10^{-2}\Gamma_{\|}\cos(3\times 10^9 t - 8x + 6z)\\ &= -\hat{\mathbf{y}}\,0.6\times 10^{-2}\cos(3\times 10^9 t - 8x + 6z) \text{ A/m}.\end{aligned}$$

(d) In medium 2,

$$k_2 = k_1\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 10\sqrt{9} = 30 \text{ rad/m},$$

$$\theta_{\mathrm{t}} = \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\theta_{\mathrm{i}}\right] = \sin^{-1}\left[\frac{1}{3}\sin 53.13^\circ\right] = 15.47^\circ,$$

and the exponent of $\mathbf{E}^{\mathrm{t}}$ and $\mathbf{H}^{\mathrm{t}}$ is

$$-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}}) = -j30(x\sin 15.47^\circ + z\cos 15.47^\circ) = -j(8x + 28.91z).$$

Hence,

$$\begin{aligned}\widetilde{\mathbf{E}}^{\mathrm{t}} &= (\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{t}})E_0^{\mathrm{i}}\tau_{\|}e^{-j(8x+28.91z)}\\ &= (\hat{\mathbf{x}}\,0.96 - \hat{\mathbf{z}}\,0.27)\,2\times 10^{-2}\times 377\times 0.44\,e^{-j(8x+28.91z)}\\ &= (\hat{\mathbf{x}}\,3.18 - \hat{\mathbf{z}}\,0.90)\,e^{-j(8x+28.91z)},\\ \widetilde{\mathbf{H}}^{\mathrm{t}} &= \hat{\mathbf{y}}\,\frac{E_0^{\mathrm{i}}\tau_{\|}}{\eta_2}\,e^{-j(8x+28.91z)}\\ &= \hat{\mathbf{y}}\,2.64\times 10^{-2}\,e^{-j(8x+28.91z)},\\ \mathbf{E}^{\mathrm{t}} &= \mathfrak{Re}\{\widetilde{\mathbf{E}}^{\mathrm{t}}e^{j\omega t}\}\\ &= (\hat{\mathbf{x}}\,3.18 - \hat{\mathbf{z}}\,0.90)\cos(3\times 10^9 t - 8x - 28.91z) \text{ V/m},\\ \mathbf{H}^{\mathrm{t}} &= \hat{\mathbf{y}}\,2.64\times 10^{-2}\cos(3\times 10^9 t - 8x - 28.91z) \text{ A/m}.\end{aligned}$$

(e)

$$S_{\mathrm{av}}^{\mathrm{t}} = \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} = \frac{|H_0^{\mathrm{t}}|^2}{2}\eta_2 = \frac{(2.64\times 10^{-2})^2}{2}\times 125.67 = 44 \text{ mW/m}^2.$$

Problem 8.30 Natural light is randomly polarized, which means that, on average, half the light energy is polarized along any given direction (in the plane orthogonal to the direction of propagation) and the other half of the energy is polarized along the direction orthogonal to the first polarization direction. Hence, when treating natural light incident upon a planar boundary, we can consider half of its energy to be in the form of parallel-polarized waves and the other half as perpendicularly polarized waves. Determine the fraction of the incident power reflected by the planar surface of a piece of glass with $n = 1.5$ when illuminated by natural light at 70°.

Solution: Assume the incident power is 1 W. Hence:

Incident power with parallel polarization $= 0.5$ W,

Incident power with perpendicular polarization $= 0.5$ W.

$\varepsilon_2/\varepsilon_1 = (n_2/n_1)^2 = n^2 = 1.5^2 = 2.25$. Equations (8.60) and (8.68) give

$$\Gamma_\perp = \frac{\cos 70^\circ - \sqrt{2.25 - \sin^2 70^\circ}}{\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = -0.55,$$

$$\Gamma_\parallel = \frac{-2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}}{2.25\cos 70^\circ + \sqrt{2.25 - \sin^2 70^\circ}} = 0.21.$$

Reflected power with parallel polarization $= 0.5\,(\Gamma_\parallel)^2$
$= 0.5\,(0.21)^2 = 22$ mW,

Reflected power with perpendicular polarization $= 0.5\,(\Gamma_\perp)^2$
$= 0.5\,(0.55)^2 = 151.3$ mW.

Total reflected power $= 22 + 151.3 = 173.3$ mW, or 17.33%..

Problem 8.43 A waveguide, with dimensions $a = 1$ cm and $b = 0.7$ cm, is to be used at 20 GHz. Determine the wave impedance for the dominant mode when

(a) the guide is empty, and

(b) the guide is filled with polyethylene (whose $\varepsilon_r = 2.25$).

Solution:

For the TE_{10} mode,

$$f_{10} = \frac{u_{p_0}}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}}.$$

When empty,

$$f_{10} = \frac{3\times 10^8}{2\times 10^{-2}} = 15 \text{ GHz}.$$

When filled with polyethylene, $f_{10} = 10$ GHz.

According to Eq. (8.111),

$$Z_{\text{TE}} = \frac{\eta}{\sqrt{1-(f_{10}/f)^2}} = \frac{\eta_0}{\sqrt{\varepsilon_r}\sqrt{1-(f_{10}/f)^2}}.$$

When empty,

$$Z_{\text{TE}} = \frac{377}{\sqrt{1-(15/20)^2}} = 570\ \Omega.$$

When filled,

$$Z_{\text{TE}} = \frac{377}{\sqrt{2.25}\sqrt{1-(10/20)^2}} = 290\ \Omega.$$

Problem 8.44 A narrow rectangular pulse superimposed on a carrier with a frequency of 9.5 GHz was used to excite all possible modes in a hollow guide with $a = 3$ cm and $b = 2.0$ cm. If the guide is 100 m in length, how long will it take each of the excited modes to arrive at the receiving end?

Solution:
With $a = 3$ cm, $b = 2$ cm, and $u_{\mathrm{p_0}} = c = 3 \times 10^8$ m/s, application of Eq. (8.106) leads to:

$$\begin{aligned} f_{10} &= 5 \text{ GHz} \\ f_{01} &= 7.5 \text{ GHz} \\ f_{11} &= 9.01 \text{ GHz} \\ f_{20} &= 10 \text{ GHz} \end{aligned}$$

Hence, the pulse with a 9.5-GHz carrier can excite the top three modes. Their group velocities can be calculated with the help of Eq. (8.114),

$$u_{\mathrm{g}} = c\sqrt{1 - (f_{mn}/f)^2},$$

which gives:

$$u_{\mathrm{g}} = \begin{cases} 0.85c = 2.55 \times 10^8 \text{ m/s}, & \text{for TE}_{10} \\ 0.61c = 1.84 \times 10^8 \text{ m/s}, & \text{for TE}_{01} \\ 0.32c = 0.95 \times 10^8 \text{ m/s}, & \text{for TE}_{11} \text{ and TM}_{11} \end{cases}$$

Travel time associated with these modes is:

$$T = \frac{d}{u_{\mathrm{g}}} = \frac{100}{u_{\mathrm{g}}} = \begin{cases} 0.39\ \mu\text{s}, & \text{for TE}_{10} \\ 0.54\ \mu\text{s}, & \text{for TE}_{01} \\ 1.05\ \mu\text{s}, & \text{for TE}_{11} \text{ and TM}_{11}. \end{cases}$$

Problem 9.2 A 1-m–long dipole is excited by a 1-MHz current with an amplitude of 12 A. What is the average power density radiated by the dipole at a distance of 5 km in a direction that is 45° from the dipole axis?

Solution: At 1 MHz, $\lambda = c/f = 3 \times 10^8/10^6 = 300$ m. Hence $l/\lambda = 1/300$, and therefore the antenna is a Hertzian dipole. From Eq. (9.12),

$$\begin{aligned} S(R,\theta) &= \left(\frac{\eta_0 k^2 I_0^2 l^2}{32\pi^2 R^2}\right)\sin^2\theta \\ &= \frac{120\pi \times (2\pi/300)^2 \times 12^2 \times 1^2}{32\pi^2 \times (5 \times 10^3)^2}\sin^2 45^\circ = 1.51 \times 10^{-9} \quad (\text{W/m}^2). \end{aligned}$$

Problem 9.4 Repeat Problem 9.3 for an antenna with

$$F(\theta,\phi) = \begin{cases} \sin^2\theta\cos^2\phi & \text{for } 0 \le \theta \le \pi \\ & \text{and } -\pi/2 \le \phi \le \pi/2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: The direction of maximum radiation is the $+\hat{\mathbf{x}}$-axis (where $\theta = \pi/2$ and $\phi = 0$). From Eq. (9.23),

$$\begin{aligned} D &= \frac{4\pi}{\iint_{4\pi} F\, d\Omega} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \sin^2\theta\cos^2\phi\sin\theta\, d\theta\, d\phi} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2}\cos^2\phi\, d\phi \int_0^{\pi}\sin^3\theta\, d\theta} \\ &= \frac{4\pi}{\int_{-\pi/2}^{\pi/2}\frac{1}{2}(1+\cos 2\phi)\, d\phi \int_{-1}^{1}(1-x^2)\, dx} \\ &= \frac{4\pi}{\frac{1}{2}\left(\phi+\frac{1}{2}\sin 2\phi\right)\Big|_{-\pi/2}^{\pi/2}(x-x^3/3)\Big|_{-1}^{1}} = \frac{4\pi}{\frac{1}{2}\pi(4/3)} = 6 = 7.8 \text{ dB}, \\ \Omega_{\text{p}} &= \frac{4\pi \text{ sr}}{D} = \frac{4\pi \text{ sr}}{6} = \frac{2}{3}\pi \quad \text{(sr)}. \end{aligned}$$

In the x-z plane, $\phi = 0$ and the half power beamwidth is 90°, since $\sin^2(45^\circ) = \sin^2(135^\circ) = \frac{1}{2}$.

Problem 9.5 A 2-m–long center-fed dipole antenna operates in the AM broadcast band at 1 MHz. The dipole is made of copper wire with a radius of 1 mm.

(a) Determine the radiation efficiency of the antenna.

(b) What is the antenna gain in decibels?

(c) What antenna current is required so that the antenna will radiate 80 W, and how much power will the generator have to supply to the antenna?

Solution:

(a) Following Example 9-3, $\lambda = c/f = (3 \times 10^8 \text{ m/s})/(10^6 \text{ Hz}) = 300$ m. As $l/\lambda = (2 \text{ m})/(300 \text{ m}) = 6.7 \times 10^{-3}$, this antenna is a short (Hertzian) dipole. Thus, from respectively Eqs. (9.35), (9.32), and (9.31),

$$R_{\text{rad}} = 80\pi^2 \left(\frac{l}{\lambda}\right)^2 = 80\pi^2 (6.7 \times 10^{-3})^2 = 35 \quad (\text{m}\Omega),$$

$$R_{\text{loss}} = \frac{l}{2\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{2 \text{ m}}{2\pi(10^{-3} \text{ m})}\sqrt{\frac{\pi(10^6 \text{ Hz})(4\pi \times 10^{-7} \text{ H/m})}{5.8 \times 10^7 \text{ S/m}}} = 83 \quad (\text{m}\Omega),$$

$$\xi = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}} = \frac{35 \text{ m}\Omega}{35 \text{ m}\Omega + 83 \text{ m}\Omega} = 29.7\%.$$

(b) From Example 9-2, a Hertzian dipole has a directivity of 1.5. The gain, from Eq. (9.29), is $G = \xi D = 0.297 \times 1.5 = 0.44 = -3.5$ dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2(80 \text{ W})}{35 \text{ m}\Omega}} = 67.6 \text{ A}$$

and from Eq. (9.31),

$$P_t = \frac{P_{\text{rad}}}{\xi} = \frac{80 \text{ W}}{0.297} = 269 \text{ W}.$$

Problem 9.11 Repeat Problem 9.5 for a 1-m–long half-wave dipole that operates in the FM/TV broadcast band at 150 MHz.

Solution:

(a) Following Example 9-3,

$$\lambda = c/f = (3 \times 10^8 \text{ m/s})/(150 \times 10^6 \text{ Hz}) = 2 \text{ m}.$$

As $l/\lambda = (1 \text{ m})/(2 \text{ m}) = \frac{1}{2}$, this antenna is a half-wave dipole. Thus, from Eq. (9.48), (9.32), and (9.31),

$$R_{\text{rad}} = 73\ \Omega,$$

$$R_{\text{loss}} = \frac{l}{2\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = \frac{1 \text{ m}}{2\pi(10^{-3} \text{ m})}\sqrt{\frac{\pi(150 \times 10^6 \text{ Hz})(4\pi \times 10^{-7} \text{ H/m})}{5.8 \times 10^7 \text{ S/m}}} = 0.5\ \Omega,$$

$$\xi = \frac{R_{\text{rad}}}{R_{\text{rad}} + R_{\text{loss}}} = \frac{73\ \Omega}{73\ \Omega + 0.5\ \Omega} = 99.3\%.$$

(b) From Eq. (9.47), a half-wave dipole has a directivity of 1.64. The gain, from Eq. (9.29), is $G = \xi D = 0.993 \times 1.64 = 1.63 = 2.1$ dB.

(c) From Eq. (9.30a),

$$I_0 = \sqrt{\frac{2P_{\text{rad}}}{R_{\text{rad}}}} = \sqrt{\frac{2(80 \text{ W})}{73\ \Omega}} = 1.48 \text{ A},$$

and from Eq. (9.31),

$$P_t = \frac{P_{\text{rad}}}{\xi} = \frac{80 \text{ W}}{0.993} = 80.4 \text{ W}.$$

Problem 9.15 For a dipole antenna of length $l = 3\lambda/2$,

(a) Determine the directions of maximum radiation.

(b) Obtain an expression for $S_{\max}$

(c) Generate a plot of the normalized radiation pattern $F(\theta)$.

(d) Compare your pattern with that shown in Fig. 9-17(c).

Solution:

(a) From Eq. (9.56), $S(\theta)$ for an arbitrary length dipole is given by

$$S(\theta) = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{\pi l}{\lambda}\cos\theta\right) - \cos\left(\frac{\pi l}{\lambda}\right)}{\sin\theta}\right]^2.$$

For $l = 3\lambda/2,\ S(\theta)$ becomes

$$S(\theta) = \frac{15I_0^2}{\pi R^2}\left[\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2.$$

Solving for the directions of maximum radiation numerically yields two maximum directions of radiation given by

$$\theta_{\max_1} = 42.6^\circ, \qquad \theta_{\max_2} = 137.4^\circ.$$

(b) From the numerical results, it was found that $S(\theta) = 15I_0^2/(\pi R^2)(1.96)$ at $\theta_{\max}$. Thus,

$$S_{\max} = \frac{15I_0^2}{\pi R^2}(1.96).$$

(c) The normalized radiation pattern is given by Eq. (9.13) as

$$F(\theta) = \frac{S(\theta)}{S_{\max}}.$$

Using the expression for $S(\theta)$ from part (a) with the value of $S_{\max}$ found in part (b),

$$F(\theta) = \frac{1}{1.96}\left[\frac{\cos\left(\frac{3\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2.$$

The normalized radiation pattern is shown in Fig. P9.15, which is identical to that shown in Fig. 9.17(c).

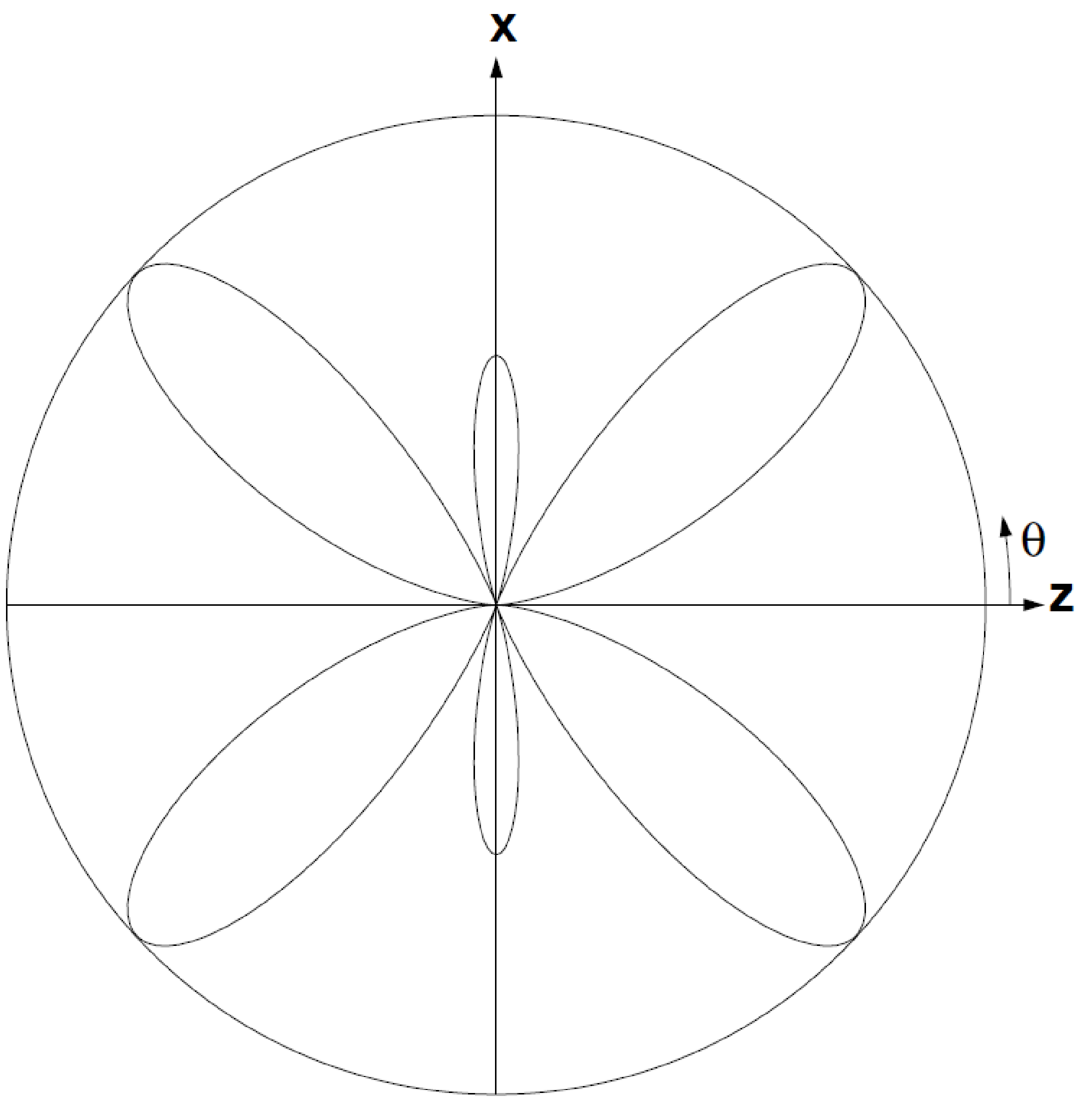

Figure P9.15: Radiation pattern of dipole of length $3\lambda/2$.